# Lesson 5. Markov Chains – $n$-Step Probabilities

## 0 Warm up – The Case of The Defective Detective

**Example 1.** Quality control engineers at KRN Corporation are monitoring the performance of a manufacturing system that produces an electronic component. Components are inspected in the sequence they are produced. The engineers believe that there is some dependence between successively produced components, and so they model whether a component is acceptable or defective by a Markov chain with states $\mathcal{M} = \{1, 2\}$ (1 = acceptable, 2 = defective), and transition probability matrix and initial state vector

$$\mathbf{P} = \begin{bmatrix} 0.995 & 0.005 \\ 0.495 & 0.505 \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} 0.96 \\ 0.04 \end{bmatrix}$$

a. What does $p_{21} = 0.495$ represent in the context of this problem?
b. Draw the transition probability diagram for this Markov chain.
c. What is the probability of the sequence of states 1, 1, 2, 2?

## 1 $n$-step transition probabilities

- Consider a Markov chain with states $\mathcal{M} = \{1, \dots, m\}$
- The $n$-**step transition probability** $p_{ij}^{(n)}$ from state $i$ to state $j$:
  - In other words, $p_{ij}^{(n)} = \Pr\{i \to j \text{ in } n \text{ steps}\}$
- Let $\mathbf{P}^{(n)}$ be the $m \times m$ matrix with elements $p_{ij}^{(n)}$
- We can compute $n$-step transition probabilities using:

**Example 2.** For the Defective Detective case, what is the probability that the third component is defective, given that the first one is not?

- Why does $\mathbf{P}^{(n)} = \mathbf{P}^n$? Let's sketch why this works for $n = 2$

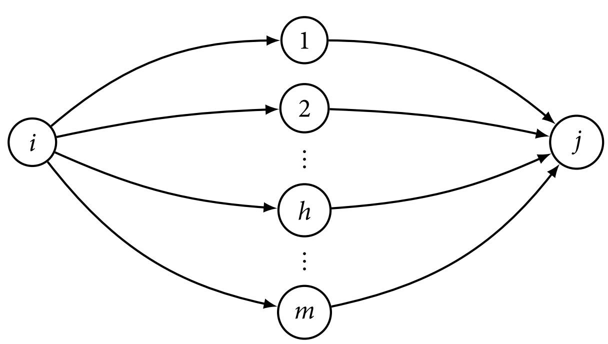
- The $i$th row of $\mathbf{P}$ looks like this:

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,m-1} & p_{1m} \\ \vdots & \vdots & & \vdots & \vdots \\ \boxed{\phantom{p_{11} \quad p_{12} \quad \cdots \quad p_{1,m-1} \quad p_{1m}}} \\ \vdots & \vdots & & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{m,m-1} & p_{mm} \end{bmatrix}$$

- The $j$th column of $\mathbf{P}$ looks like this:

$$\begin{bmatrix} p_{11} & \cdots & \boxed{\phantom{p}} & \cdots & p_{1m} \\ p_{21} & \cdots & & \cdots & p_{2m} \\ \vdots & & & & \vdots \\ p_{m-1,1} & \cdots & & \cdots & p_{m-1,m} \\ p_{m1} & \cdots & & \cdots & p_{mm} \end{bmatrix}$$

- To compute $p_{ij}^{(2)}$, we need to consider all the different ways we can go from $i$ to $j$ in 2 steps:




$$\begin{aligned} p_{ij}^{(2)} = \Pr\{i \to j \text{ in 2 steps}\} &= \sum_{h=1}^{m} \Pr\{i \to h, \text{ then } h \to j\} \\ &= \sum_{h=1}^{m} p_{ih} p_{hj} \\ &= (i\text{th row of } \mathbf{P}) \cdot (j\text{th column of } \mathbf{P}) \end{aligned}$$

- The proof for arbitrary $n$ works in a similar fashion
- Also in a similar fashion, we can derive the **Chapman-Kolmogorov equation**:

$$p_{ij}^{(n)} = \sum_{h=1}^{m} p_{ih}^{(k)} p_{hj}^{(n-k)}$$

  - In other words,

$$\Pr\{i \to j \text{ in } n \text{ steps}\} = \sum_{h=1}^{m} \Pr\{i \to h \text{ in } k \text{ steps, then } h \to j \text{ in } n-k \text{ steps}\}$$

## 2 $n$-step state probabilities

- The $n$-**step state probability** $q_j^{(n)}$ for state $j$ is

  - In other words, $q_j^{(n)} = \Pr\{x \to j \text{ in } n \text{ steps, } \underline{x \text{ chosen randomly according to initial state probabilities}}\}$

- Let $\mathbf{q}^{(n)}$ be the $m \times 1$ vector with elements $q_j^{(n)}$
- We can compute $n$-step state probabilities using:

**Example 3.** For the Defective Detective case, what is the probability that the fourth component is defective?

- Why does $\mathbf{q}^{(n)\top} = \mathbf{q}^\top \mathbf{P}^n$?

$$\begin{aligned} q_j^{(n)} = \Pr\{S_n = j\} &= \sum_{i=1}^{m} \Pr\{S_n = j \mid S_0 = i\} \Pr\{S_0 = i\} \\ &= \sum_{i=1}^{m} q_i p_{ij}^{(n)} \\ &= \mathbf{q} \cdot (j\text{th column of } \mathbf{P}^n) \end{aligned}$$

## 3 First passage probabilities

- Let $\mathcal{A}$ and $\mathcal{B}$ be two disjoint subsets of the state space $\mathcal{M}$
  - e.g. $\mathcal{M} = \{1, 2, 3, 4\}$, $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$
- The **first passage probability** $f_{ij}^{(n)}$ for initial state $i \in \mathcal{A}$ and final state $j \in \mathcal{B}$ in $n$ time steps is:

- In other words,

$$f_{ij}^{(n)} = \Pr\{\text{start in } i \in \mathcal{A}, \text{ stay in states in } \mathcal{A} \text{ for } n-1 \text{ steps, end in } j \in \mathcal{B} \text{ at the } n\text{th step}\}$$

- Let $\mathbf{P}_{\mathcal{A}\mathcal{B}}$ be the submatrix of $\mathbf{P}$ whose elements are $p_{ij}$ with $i \in \mathcal{A}$ and $j \in \mathcal{B}$:
  - e.g. $\mathcal{A} = \{1, 2, 3\}$, $\mathcal{B} = \{4\}$

$$(*) \quad \mathbf{P} = \begin{bmatrix} 0 & 0.95 & 0.01 & 0.04 \\ 0.27 & 0 & 0.63 & 0.10 \\ 0.36 & 0.40 & 0 & 0.24 \\ 0.11 & 0.71 & 0.18 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_{\mathcal{A}\mathcal{B}} = \qquad \mathbf{P}_{\mathcal{A}\mathcal{A}} =$$

- Let $\mathbf{F}_{\mathcal{A}\mathcal{B}}^{(n)}$ be the $|\mathcal{A}| \times |\mathcal{B}|$ matrix whose elements are $f_{ij}^{(n)}$
- We can compute first passage probabilities using:

**Example 4.** The submarine Markov chain in Lesson 3 had the following transition probability matrix given in $(*)$ above. Recall that the submarine starts in cell 1. Compute the probability that the submarine stays in cells 1, 2, and 3 for 5 time steps, and ends up in region 4 in the 6th time step.

## 4 Next time...

- What happens in the **long run**, i.e. when the number of time steps $n$ approaches infinity?

## 5 Exercises

**Problem 1** (SMAS Exercise 6.11, modified). A food manufacturer plans to introduce a new potato chip, Box O' Spuds, into a local market that already has two strong competitors. The marketing analysts would like to forecast the long-term market share for Box O' Spuds to determine whether it is worth entering the market.

Suppose the marketing analysts formulate a Markov chain model of customer brand switching in which the state space $\mathcal{M} = \{1, 2, 3\}$ corresponds to which of the two established brands or Box O' Spuds, respectively, that a customer currently purchases. The time index is the number of bags of chips purchased. Based on market research and experience with other products, the transition probability matrix the marketing analysts anticipate is

$$\mathbf{P} = \begin{bmatrix} 0.70 & 0.28 & 0.02 \\ 0.28 & 0.70 & 0.02 \\ 0.10 & 0.10 & 0.80 \end{bmatrix}$$

a. Note that the diagonal entries of **P** are larger than the off-diagonal entries. What does this mean in the context of this problem?

b. Suppose that initially, a typical customer is equally likely to prefer one of the two existing brands. What is the probability that a typical customer prefers Box O' Spuds after he or she has bought 50 bags of chips?

c. What is the probability that a customer initially buys a bag of Brand 2 chips, buys only the two existing brands over the course of his or her next 9 bags of chips, and then purchases Box O' Spuds for his or her 11th bag of chips?

**Problem 2.** An automated guided vehicle (AGV) transports parts between three locations: the release station, the machining station, and the output buffer. The movement of the AGV can be described as making trips from location to location based on requests to move parts. Consider a Markov chain in which the states 1, 2, 3 correspond to the release station, machining station, and output buffer, respectively, and each time step corresponds to one trip of the AGV. The transition probability matrix for the AGV is:

$$\mathbf{P} = \begin{bmatrix} 1/5 & 3/5 & 1/5 \\ 1/4 & 1/2 & 1/4 \\ 1/5 & 2/5 & 2/5 \end{bmatrix}$$

a. Draw the transition probability diagram for this Markov chain.

b. Suppose the AGV is at the release station. What is the probability that it will be back at the release station in 3 trips?

c. Suppose at the beginning of the day, the AGV is equally likely to be at any of the three locations. What is the probability that it will be at the output buffer in 3 trips?

d. Suppose the AGV is currently at the machining station. What is the probability that the AGV then travels between the release station and the machining station for 4 trips, and then finally visits the output buffer in the 5th trip?

**Problem 3.** The law firm of Primal and Dual employs three types of lawyers: junior lawyers, senior lawyers, and partners. Some of these lawyers eventually leave as non-partners, others leave as partners.

Consider a Markov chain that models the career path of a lawyer at Primal and Dual with five states: Junior (1), Senior (2), Partner (3), Leave as non-partner (4), and Leave as partner (5). Each time step represents one year. The transition probability matrix and initial state probability vector is

$$\mathbf{P} = \begin{bmatrix} 0.80 & 0.15 & 0 & 0.05 & 0 \\ 0.05 & 0.65 & 0.20 & 0.10 & 0 \\ 0 & 0 & 0.95 & 0 & 0.05 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0.75 \\ 0.20 \\ 0.05 \\ 0 \\ 0 \end{bmatrix}$$

Note that a senior lawyer can be demoted to a junior lawyer.

a. Draw the transition probability diagram for this Markov chain.
b. Note that $p_{44}$ and $p_{55}$ are equal to 1. Why does this make sense in the context of the problem?
c. Suppose that you are a junior lawyer this year. What is the probability that you are a partner 5 years from now (and still at the firm)?
d. Randomly pick a lawyer at the firm today. (You would do this using the initial state probabilities given above.) What is the probability that this randomly chosen lawyer is a partner 5 years from now (and still at the firm)?
e. Again, suppose that you are a junior lawyer this year. What is the probability that you spend the next 5 years as either a junior lawyer or senior lawyer, and then leave as a non-partner in your 7th year?